AOS 801: Advanced Tropical Meteorology

Lecture 11   Spring 2023

Weak Temperature Gradient Balance

Ángel F. Adames Corraliza
angel.adamescorraliza@wisc.edu

The gravity wave equation in isobaric coordinates

Combining the three equations yields the gravity wave equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial p}\frac{1}{\sigma}\frac{\partial^2}{\partial t \partial p}\right)\Phi' = -\nabla_h^2 \Phi'.$$

Assuming a wave solution of the form:

$$\Phi' = \hat{\Phi}\exp\left(ikx + ily + imp - i\varpi t\right)$$

Yields the following phase speed:

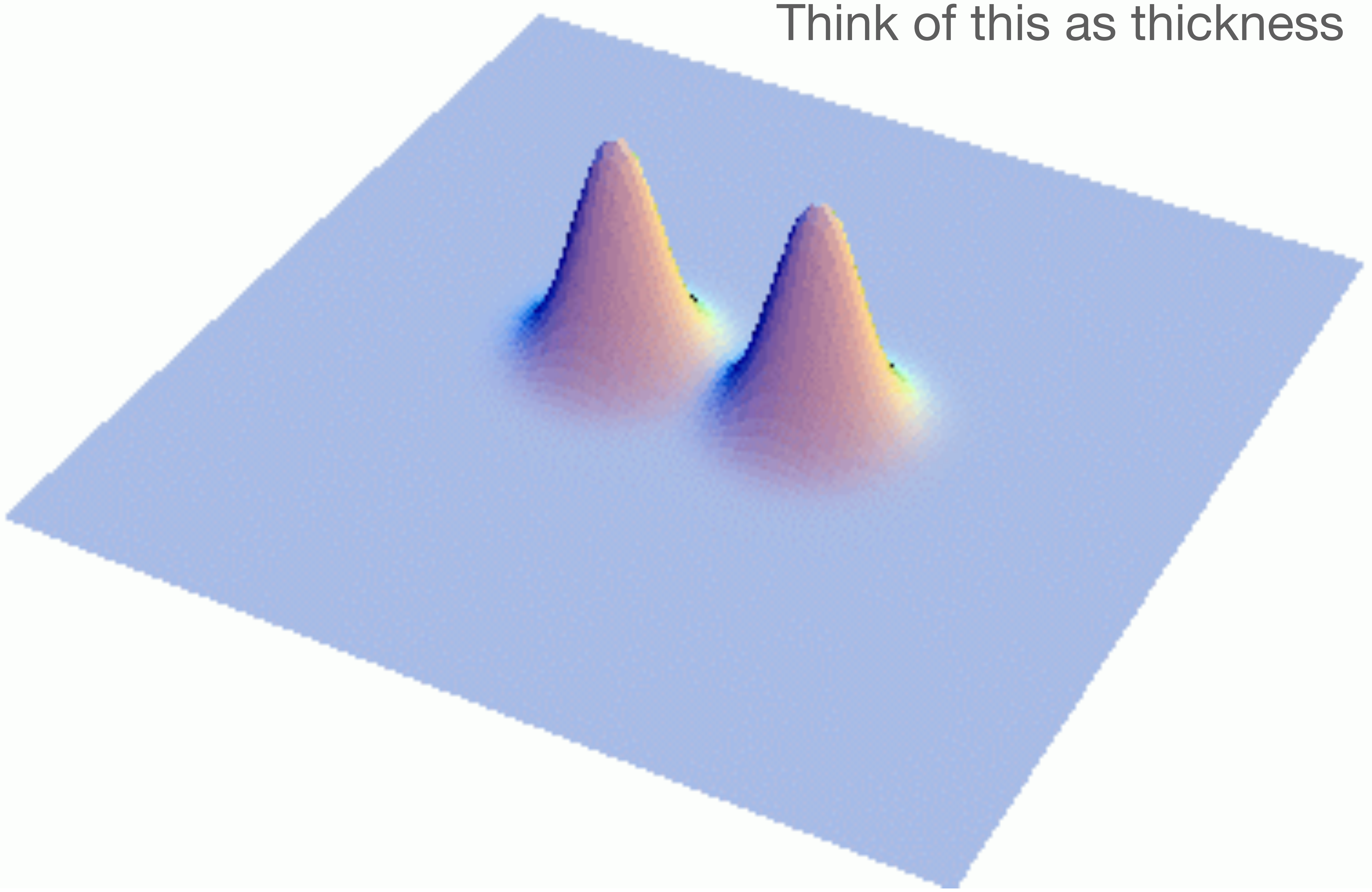
$$c = \frac{\varpi}{K} = \pm\frac{\sqrt{\sigma}}{m}, \qquad \text{where } K^2 = k^2 + l^2$$

Think about the gravity waves

Dispersion:

$$c = \frac{\varpi}{K} = \pm \frac{\sqrt{\sigma}}{m},$$

where $K^2 = k^2 + l^2$



Think about the evolution of these waves from the perspective of the thermodynamic equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi'}{\partial p}\right) = -\omega\sigma$$

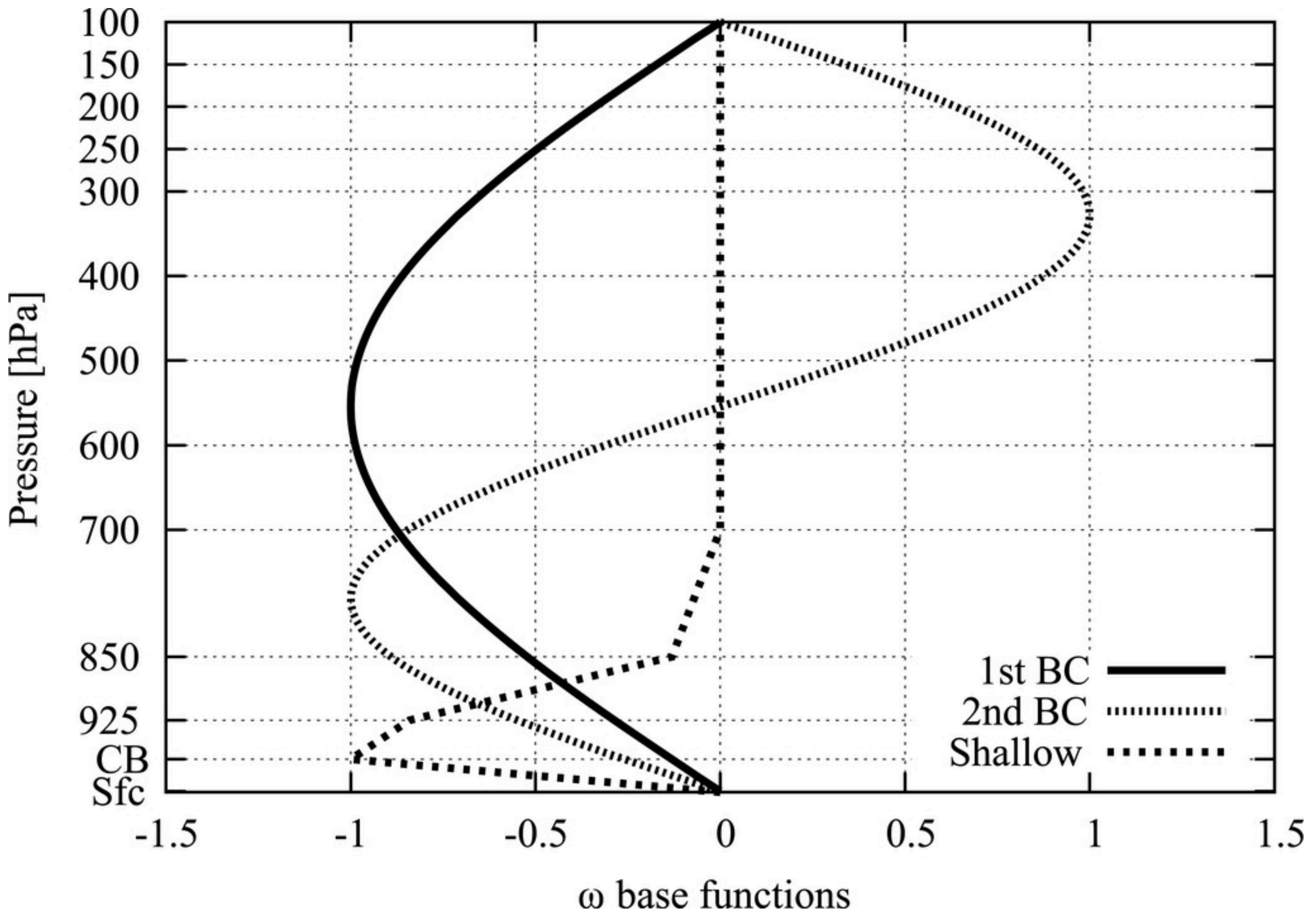
What determines the gravity wave speed?

The phase speed is :

$$c = \pm \frac{\sqrt{\sigma}}{m}$$

Plugging realistic numbers onto c yields a value of 50 m/s for the first baroclinic mode.

50 m/s = 112 mph, you can't outdrive this wave.



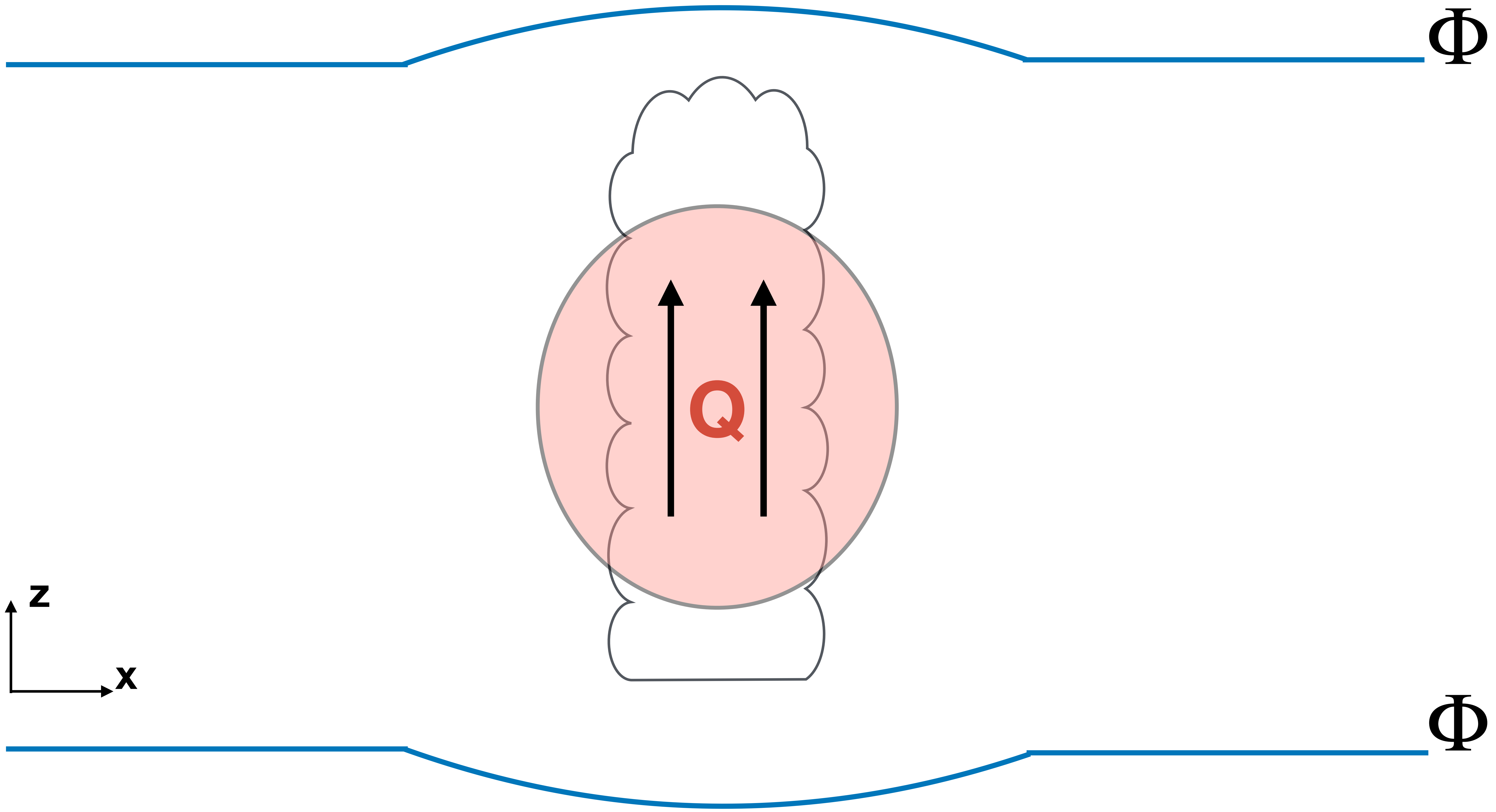
Masunaga and L'Ecuyer (2014)

Weak temperature gradient balance

A region of strong convection has a lot of latent heat release from condensation, warming up the cloud. Geopotential increases in the upper troposphere and decreases in the lower troposphere to adjust to hydrostatic balance.

$$\omega \frac{\partial \mathrm{DSE}}{\partial p} = Q_1$$

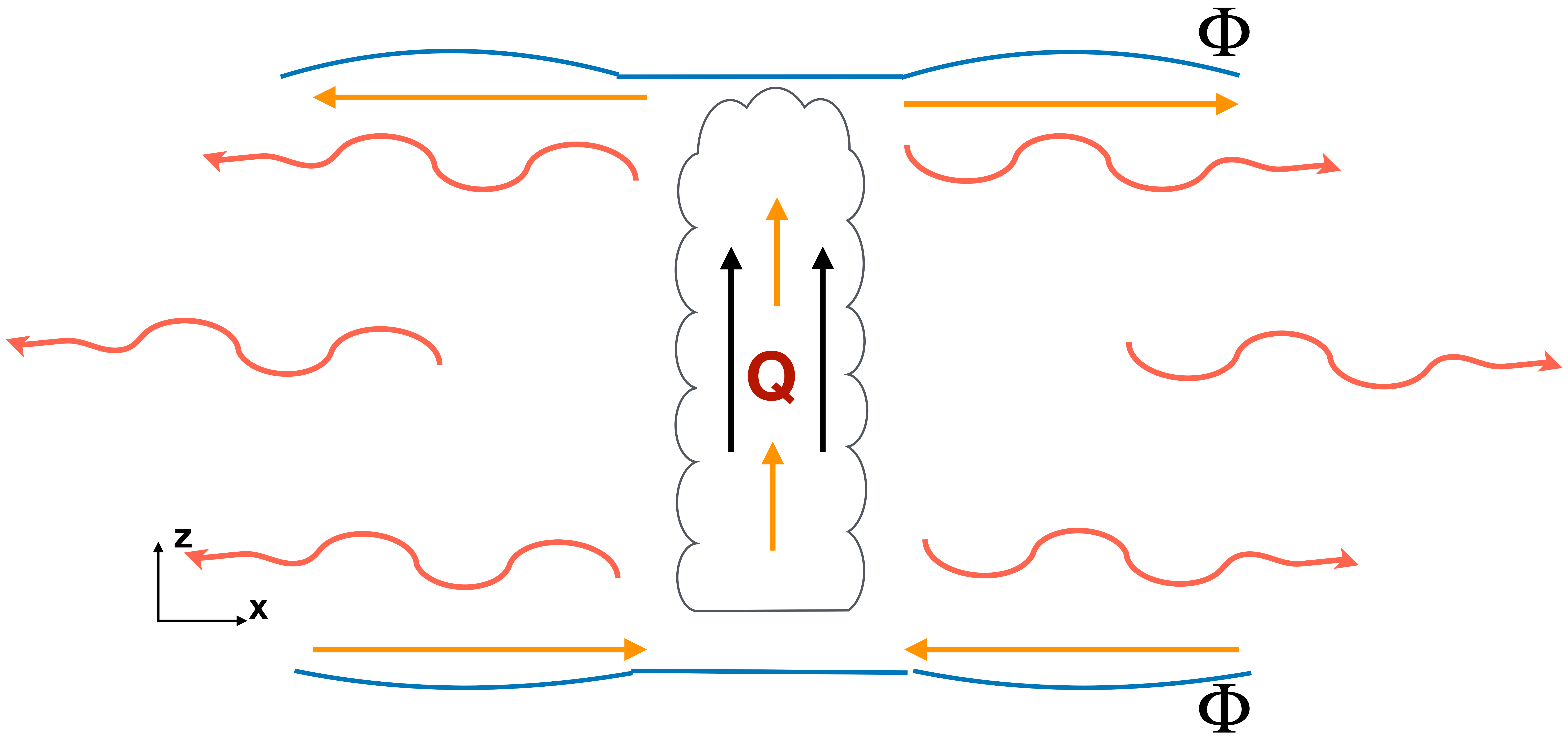
$$\mathrm{DSE} = C_p T + \Phi$$



Weak temperature gradient balance

Gravity waves develop from the convection and "smooth out" the geopotential/temperature anomalies. A *secondary* circulation develops from the gravity waves, which adds further upward motion to the convection, cooling the cloud.

$$\omega \frac{\partial \mathrm{DSE}}{\partial p} = Q_1$$



Weak temperature gradient balance

$$\omega \frac{\partial \mathrm{DSE}}{\partial p} = Q_1$$

$$\mathrm{DSE} = C_p T + \Phi$$

This process redistributes entropy (warm air has higher entropy)

Weak temperature gradient balance

Let's return to the equations that gave us gravity waves and add a heat source $\mathcal{Q}$ (i.e. a mass source). For simplicity, let's consider one dimension:

$$\frac{\partial u'}{\partial t} = -\frac{\partial \Phi'}{\partial x}$$

$$\frac{\partial \Phi'}{\partial t} + c^2 \frac{\partial u'}{\partial x} = \mathcal{Q}$$

Where we have now defined the gravity wave phase speed c a priori. The equations combine to yield:

$$\frac{\partial^2 \Phi'}{\partial t^2} - c^2 \frac{\partial^2 \Phi'}{\partial x^2} = \frac{\partial \mathcal{Q}}{\partial t}$$

Weak temperature gradient balance

The two equations can be combined to form the forced (inhomogeneous) wave equation

$$\frac{\partial^2 \Phi'}{\partial t^2} - c^2 \frac{\partial^2 \Phi'}{\partial x^2} = \frac{\partial \mathcal{Q}}{\partial t}$$

Where we can break down the heating into

$$\mathcal{Q} = F(x)H(t)$$

Where H is the Heaviside step function.

Weak temperature gradient balance

The forced wave equation has a solution in the form of a Green's function, which can be written as:

$$\Phi'(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} F(x')\delta(t')dx'dt'$$

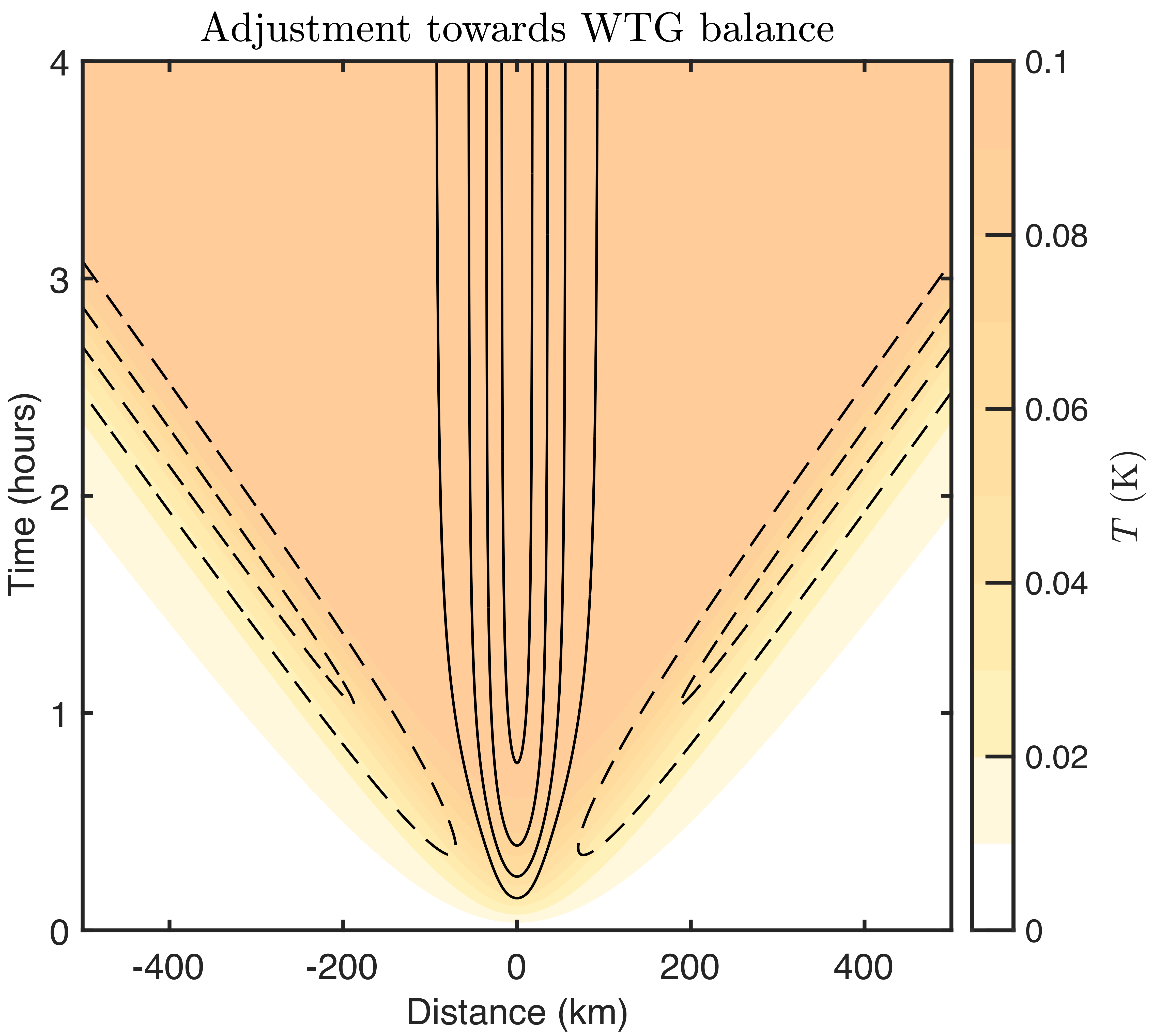
Where $\delta(t')$ is the Dirac delta function.

Note that x' and t' are different from x and t.

The forced wave equation

$$\Phi'(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} F(x')\delta(t')dx'dt'$$

The solution shows gravity waves propagating away from the heat source, warming the column adiabatically as they propagate at a phase speed c.



Contoured: vertical velocity (w)

The forced wave equation

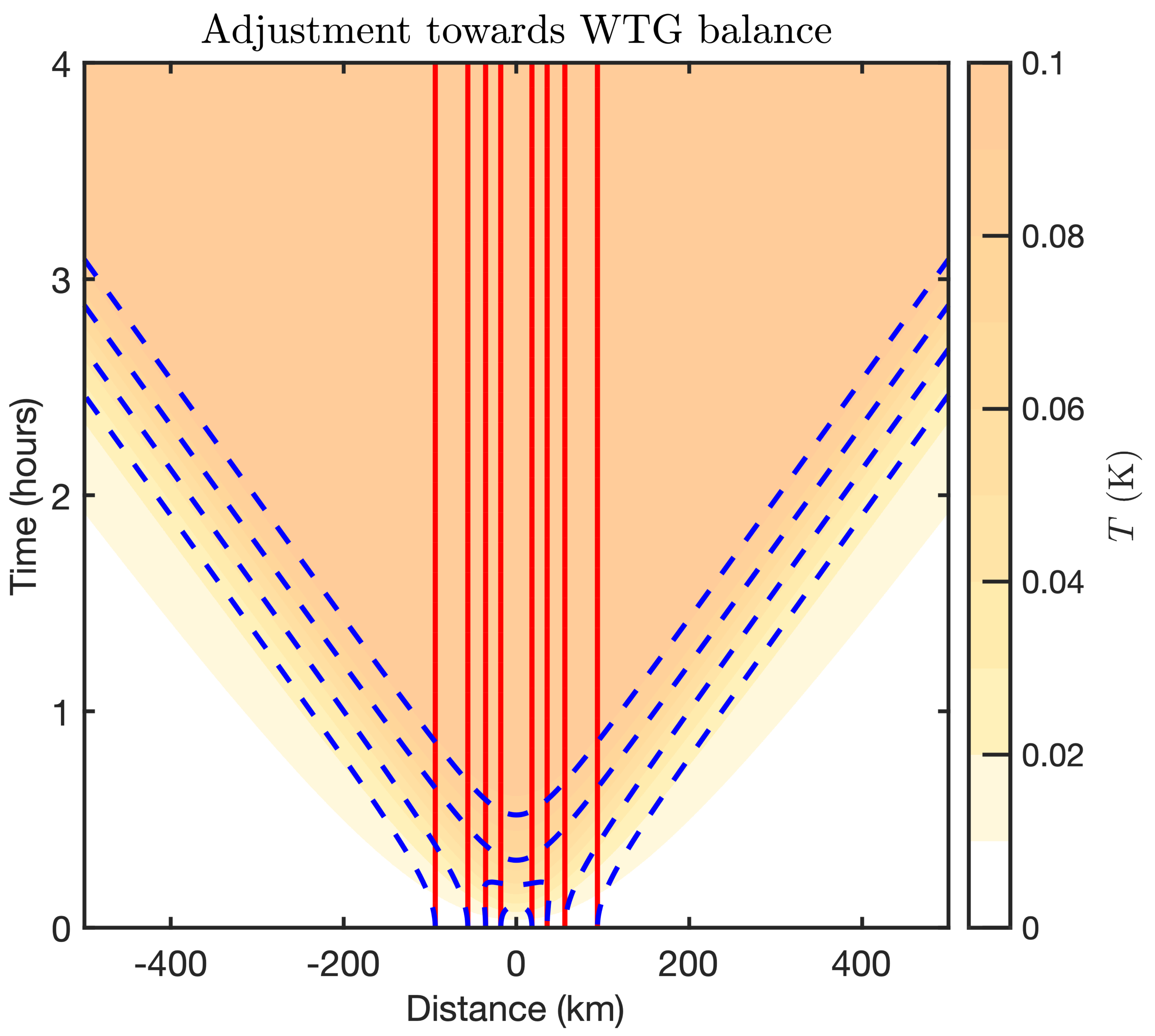
In analogy to geostrophic balance, we can define a WTG and a non-WTG vertical velocity

$$\omega = \omega_w + \omega'$$

$$\omega_w \frac{\partial \mathrm{DSE}}{\partial p} \equiv Q_1$$

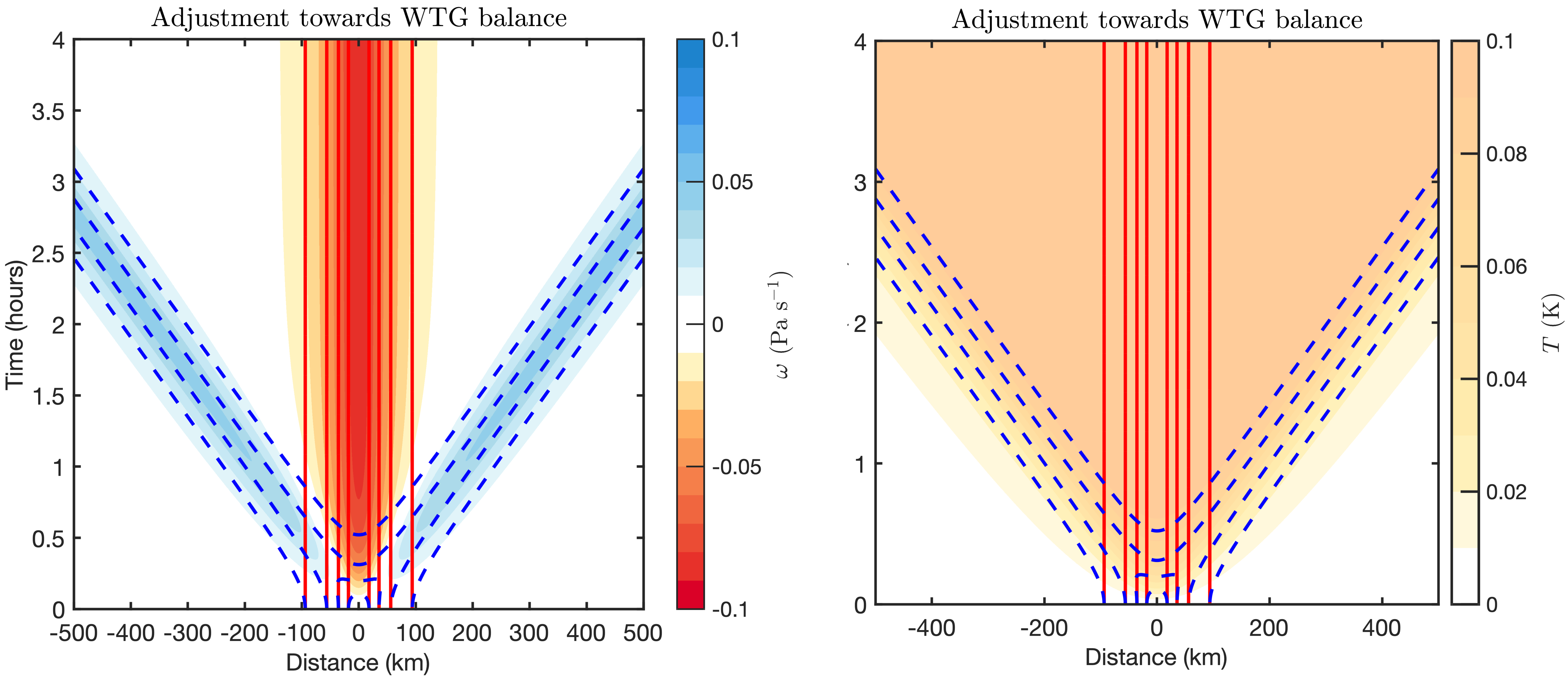
$$C_p \frac{\partial T}{\partial t} \equiv -\omega' \frac{\partial \mathrm{DSE}}{\partial p}$$

The "cloud" (Q) reaches balance very quickly!



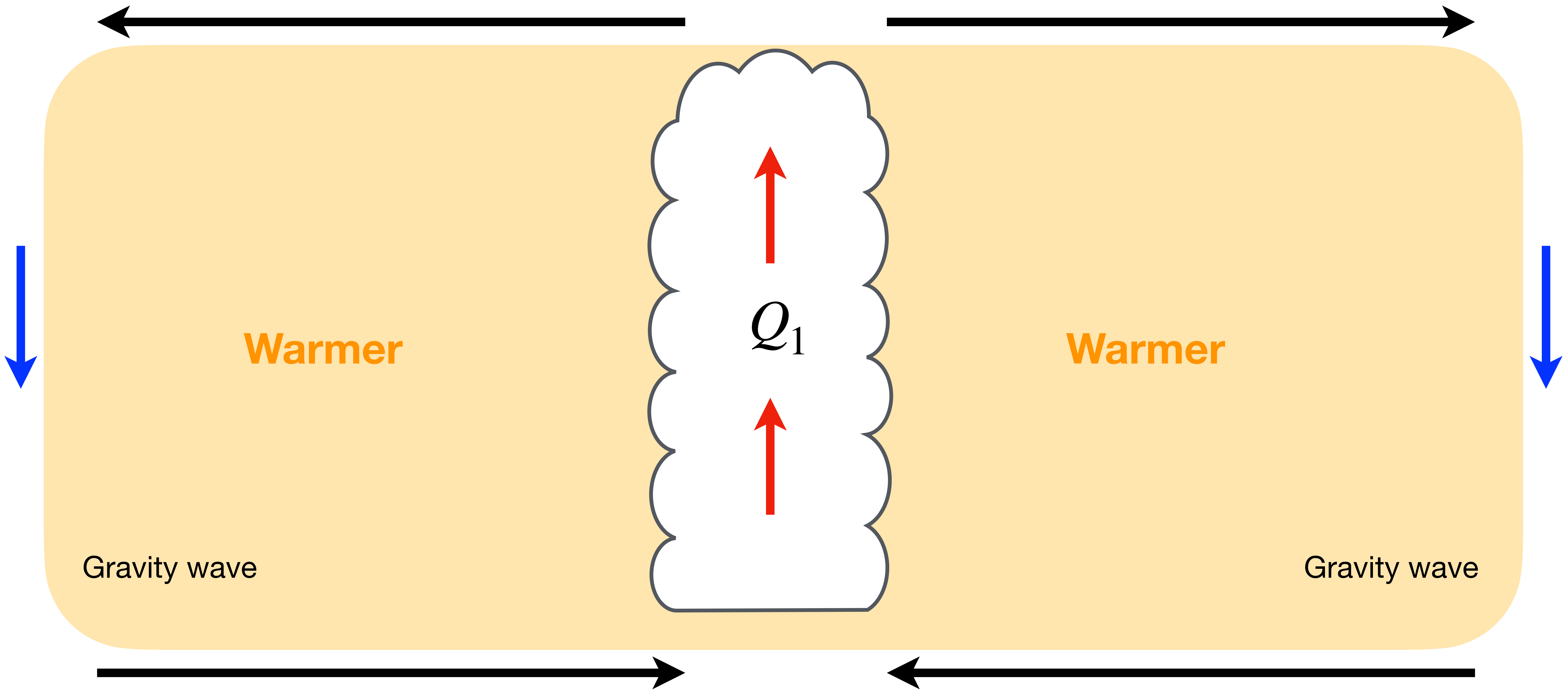
Blue: adiabatic subsidence **Red: WTG ascent**

The forced wave equation



Blue: adiabatic subsidence Red: Diabatic ascent

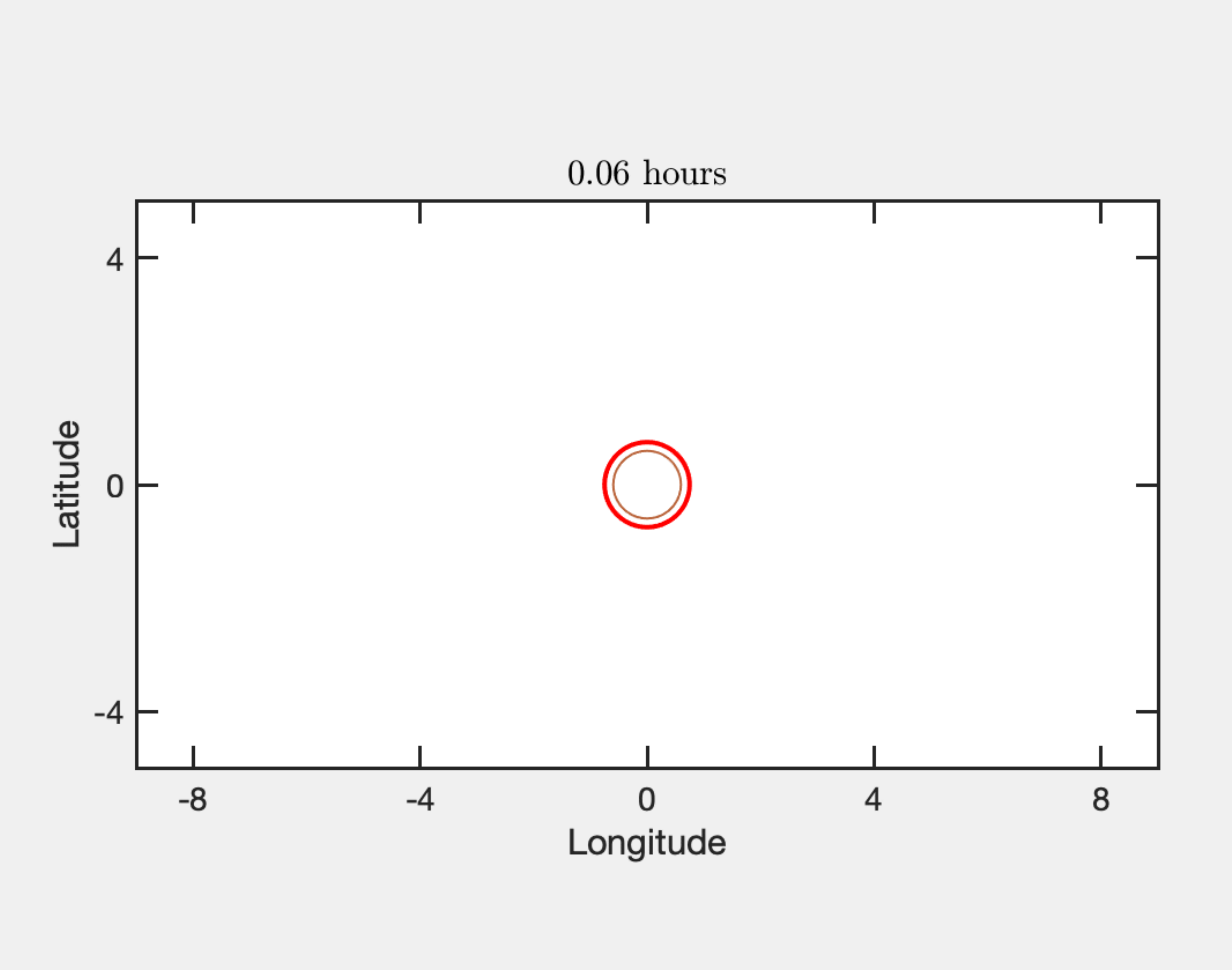
The forced wave equation



2-D Forced wave equation

$$\frac{\partial^2 \Phi'}{\partial t^2} - c^2 \nabla_h^2 \Phi' = \frac{\partial \mathcal{Q}}{\partial t}$$

The solution shows gravity waves propagating away from the heat source, warming the column as they propagate at the phase speed c.

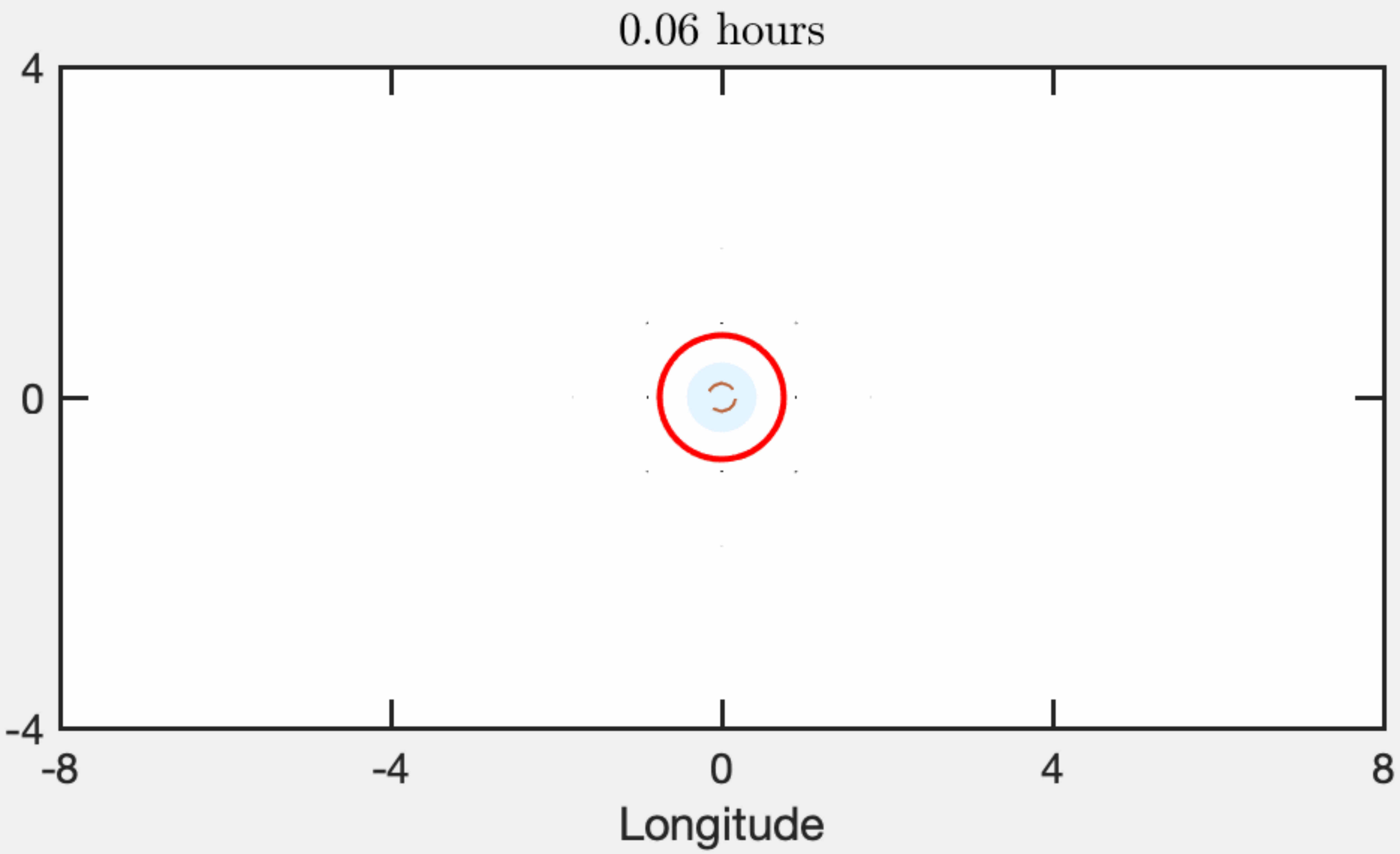


Contoured: vertical velocity (w)

2-D Forced wave equation

$$\frac{\partial^2 \Phi'}{\partial t^2} - c^2 \nabla_h^2 \Phi' = \frac{\partial \mathcal{Q}}{\partial t}$$

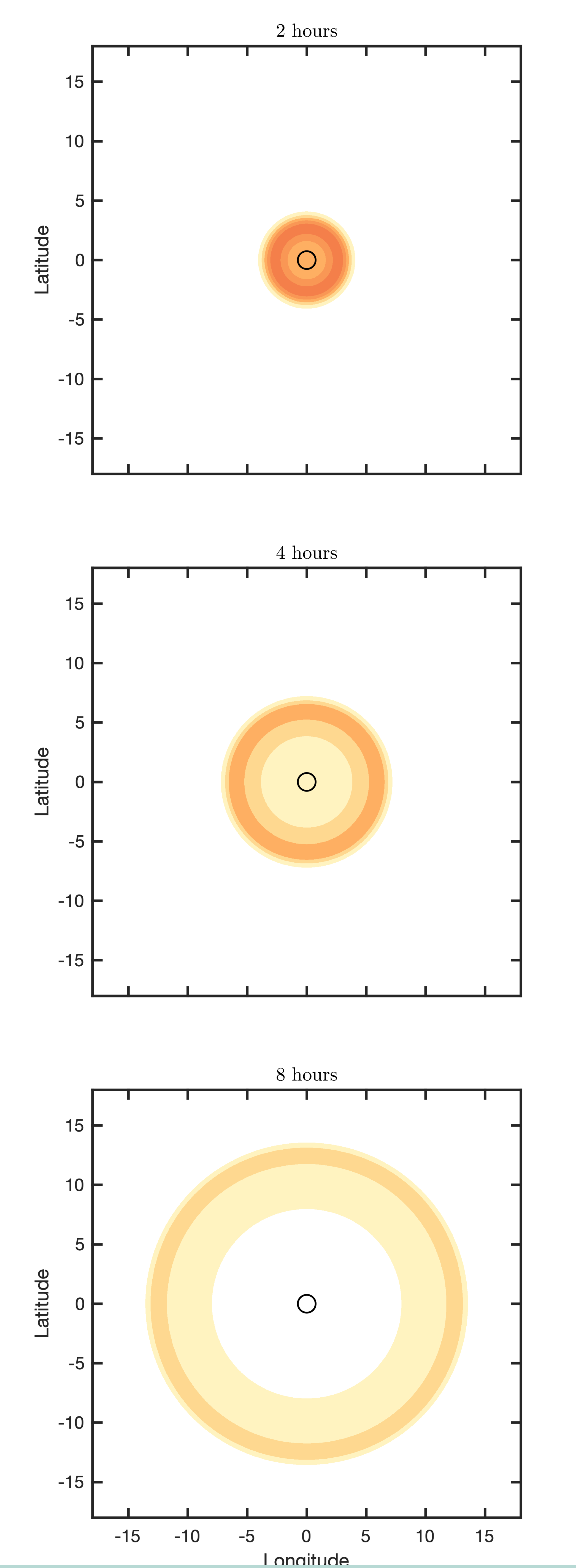
The solution shows gravity waves propagating away from the heat source, warming the column as they propagate at the phase speed c.

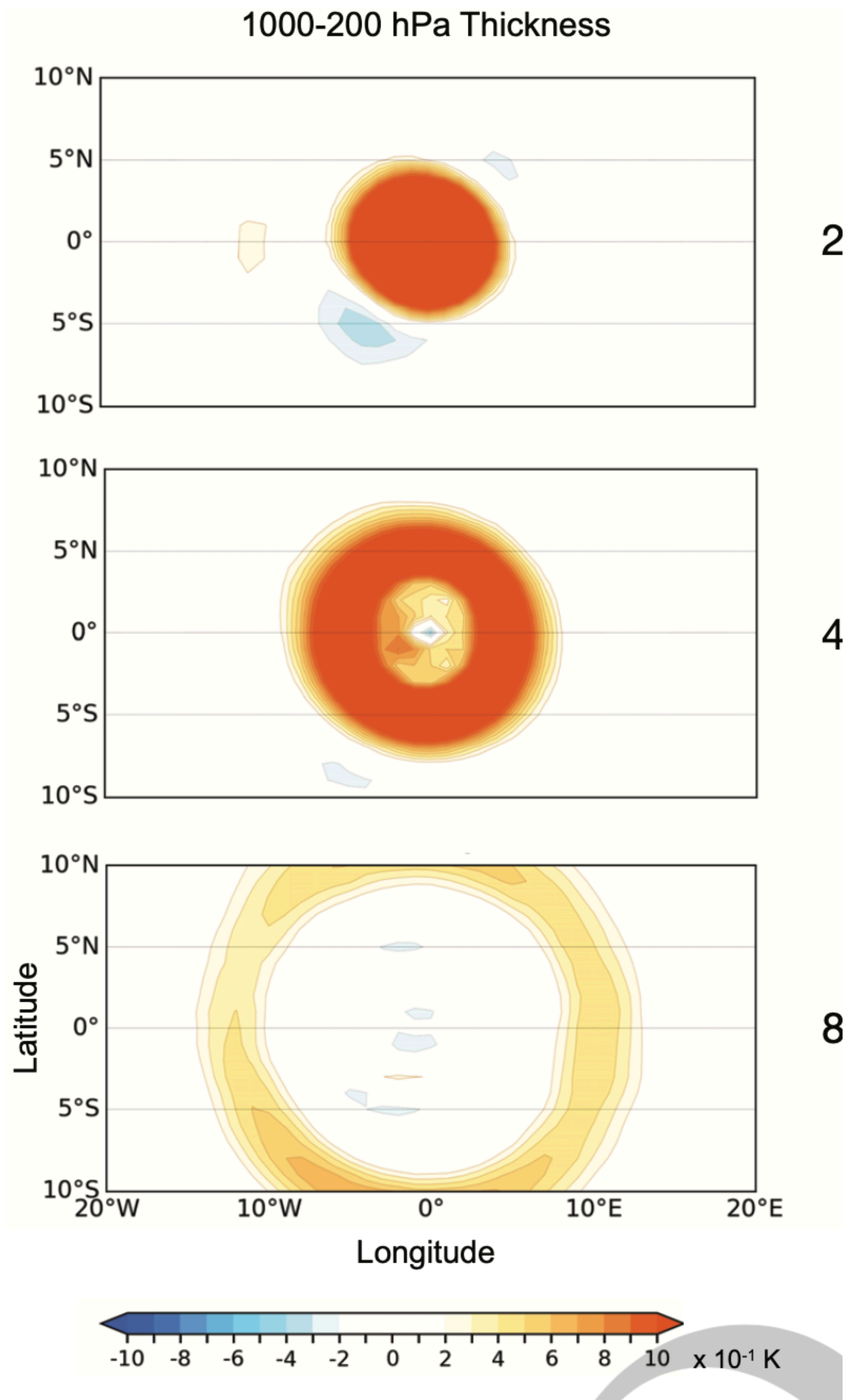


2-D Forced wave equation

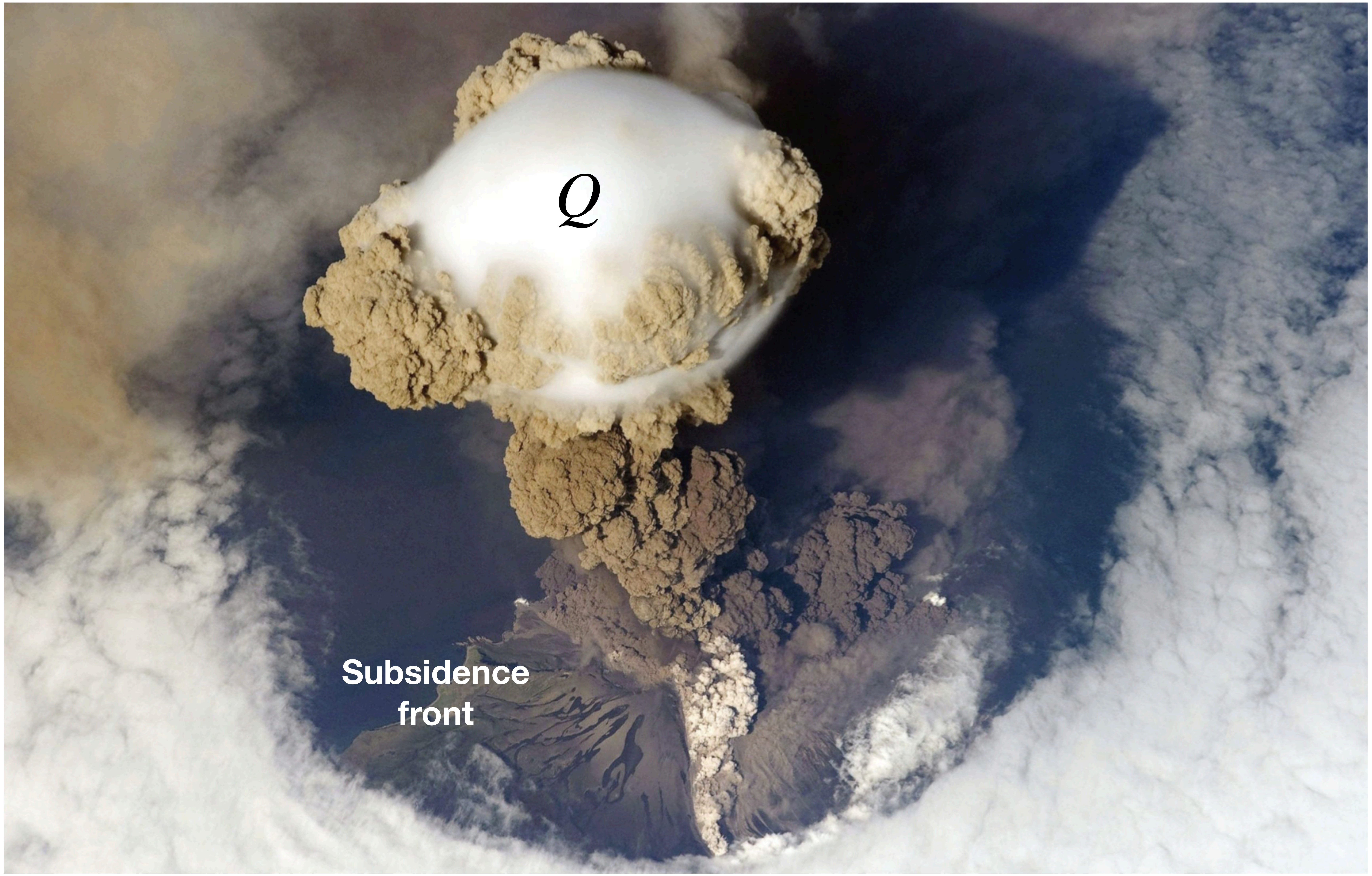
This process is clearly seen in composites based on ERA5 data.

In this instant the gravity waves follow a phase speed of the first bar-clinic mode, which is roughly 50 m/s.

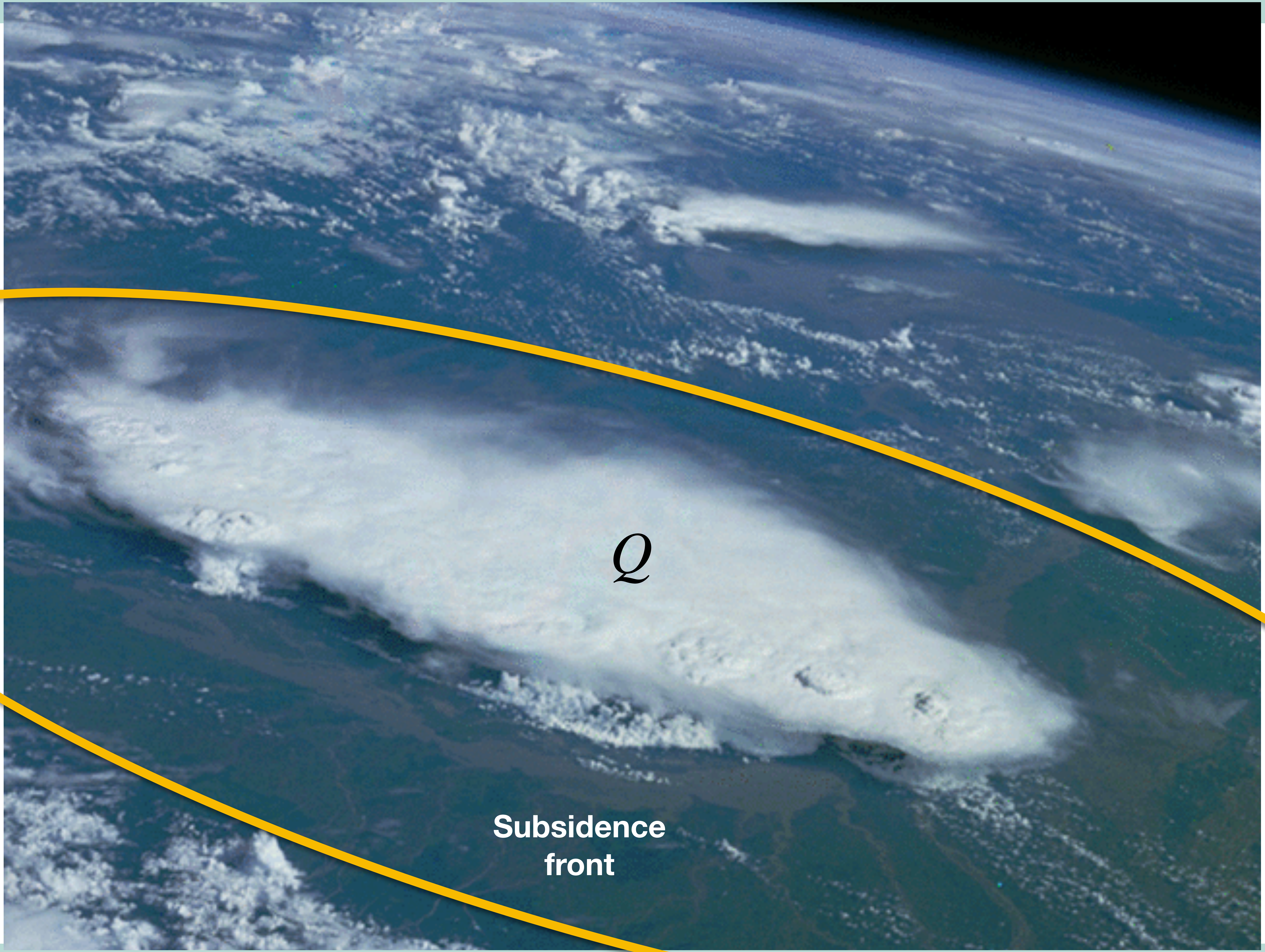




In real life



In real life



When is WTG balance valid?

$$\omega \frac{\partial \mathrm{DSE}}{\partial p} = Q_1$$

Given that there's an adjustment process associated with gravity waves, we must assume systems that are in WTG balance must evolve slowly.

But **how slowly**? We must invoke scale analysis to find out!

Basic equations

In order to understand when and why WTG is valid, we must scale the entire set of basic equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_h)\mathbf{v} + \omega \frac{\partial \mathbf{v}}{\partial p} = -f\mathbf{k} \times \mathbf{v} - \nabla_h \Phi$$

Horizontal Momentum

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

Hydrostatic

$$\frac{\partial \omega}{\partial p} = -\nabla_h \cdot \mathbf{v}$$

Mass Continuity

$$\frac{\partial C_p T}{\partial t} + \mathbf{v} \cdot \nabla_h C_p T + \omega \frac{\partial \mathrm{DSE}}{\partial p} = Q_1$$

Thermodynamic

$$\frac{\partial L_v q}{\partial t} + \mathbf{v} \cdot \nabla_h L_v q + \omega \frac{\partial L_v q}{\partial p} = -Q_2$$

Moisture

The WTG circulation

Let us decompose the wind field into a strict WTG component and a deviation from strict WTG balance.

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}'$$

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} + \omega\mathbf{k}$$

$$\mathbf{u}_w \cdot \nabla \mathrm{DSE} \equiv Q_1$$

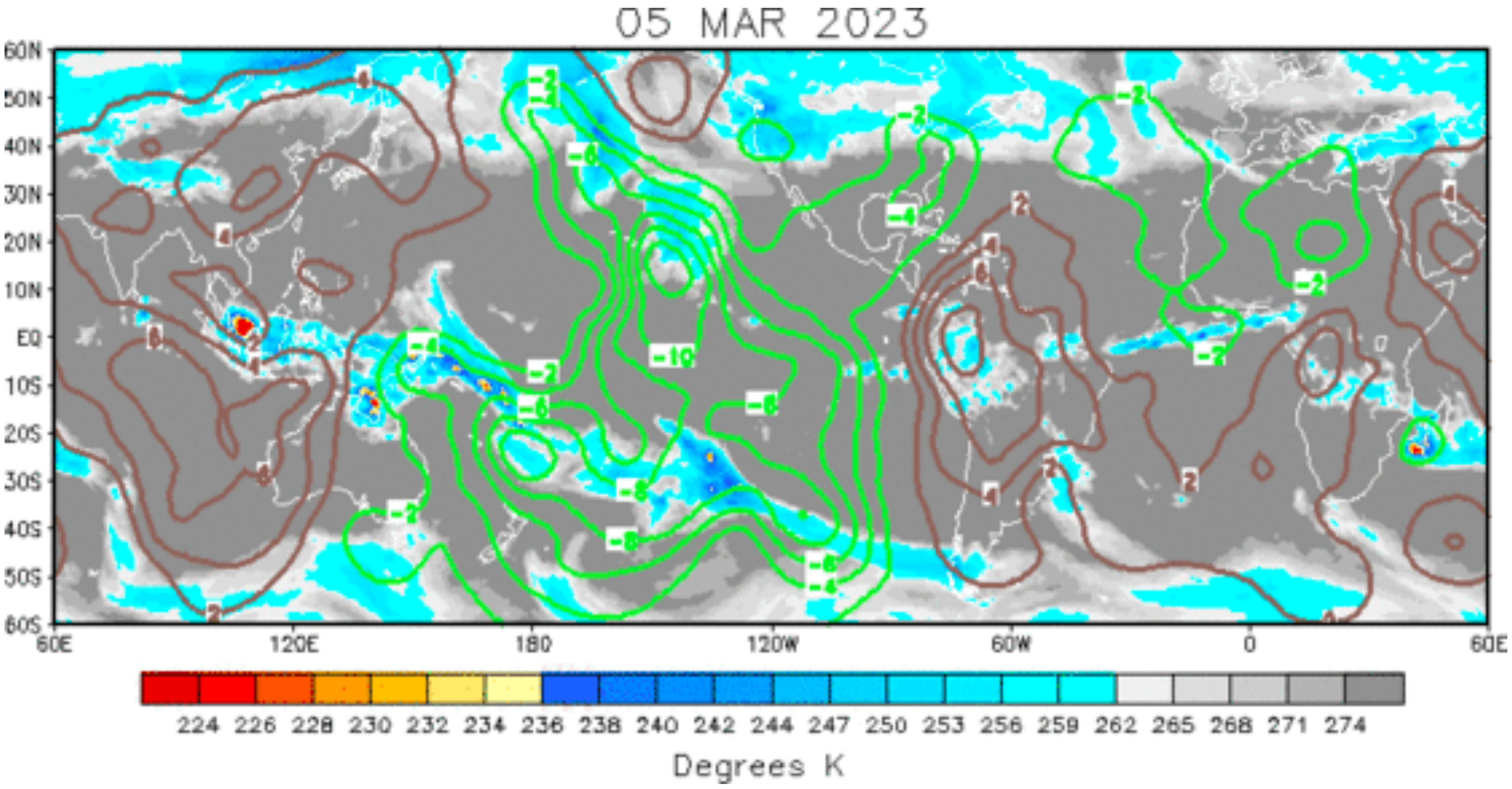
This wind is not associated with any temperature anomalies. Thus it must be purely irrotational

$$\frac{\partial \omega_w}{\partial p} = -\delta_w = -\nabla_h^2 \chi_w .$$

$$\mathbf{v}_w = \nabla_h \chi_w$$

χ_w is the **WTG velocity potential**

Velocity potential



WTG deviation

The deviation from WTG balance is defined as

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}' \qquad \mathbf{u} = u\mathbf{i} + v\mathbf{j} + \omega\mathbf{k}$$

$$\mathbf{v}' = \mathbf{k} \times \nabla_h \psi' + \nabla_h \chi'$$

Note that the non-divergent component of the wind field is due to deviations from strict WTG balance.