Let's think about entrainment:

$$\frac{DMSE_c}{Dt} = Q_e - \frac{1}{m}\left(\frac{Dm}{Dt}\right)_\varepsilon (MSE_c - MSE_e)$$

diabetic sources/sinks of MSE

Mass exchange rate due to entrainment

Dilution b.c. $MSE_e < MSE_c$ b.c. cloud is saturated

Let's assume that $Q_e = 0$ and that

$$\frac{DMSE_c}{Dt} = \cancel{\frac{\partial MSE_c}{\partial t}} + \cancel{\vec{v}\cdot\nabla_h MSE_c} + w\frac{\partial MSE_c}{\partial z}$$

MSE varies more rapidly in the vertical, even when accounting for $w << |\vec{v}|$

Expanding into MSE budget.

$$\cancel{w}\frac{\partial MSE}{\partial z} = -\frac{\cancel{w}}{m}\left(\frac{\partial m}{\partial z}\right)_\varepsilon (MSE_c - MSE_e)$$

$$\varepsilon = \frac{1}{m}\left(\frac{\partial m}{\partial z}\right)_\varepsilon = \text{entrainment rate}$$

$$\frac{\partial MSE_c}{\partial z} = -\varepsilon (MSE_c - MSE_e)$$

Entraining plume MSE budget.

What if $T_e = T_c$ Convective QE

The cloud is saturated so $MSE_c = MSE_c^*$

$$\frac{\partial MSE_c}{\partial z} = -\varepsilon L_v (q_e^* - q_e)$$

Dilution is related to the saturation deficit. $q^+ = q^* - q$

$$B = -\frac{\rho'}{\rho} g \simeq \frac{T_c - T_e}{T} = \frac{T'}{T} = \frac{dT}{T}$$

$$dT \propto dMSE^* \qquad dMSE^* = d c_p T + d\Phi + L_v dq^*$$

$$= c_p dT + L_v \frac{dq^*}{dT} dT$$

$$dMSE^* = c_p \underbrace{\left(1 + \frac{L_v}{c_p}\frac{dq^*}{dT}\right)}_{K_r} dT$$

$$dT = \frac{1}{c_p K_r} dMSE^*$$

Plug this differential onto B:

$$B = \frac{MSE^*_c - MSE^*_e}{K_r c_p T}$$

Let's integrate the plume MSE budget

$$\int_{z_B}^{z} \frac{\partial MSE^*_c}{\partial z} dz' = \int_{z_B}^{z} -\varepsilon L_v \left(q^*_c - q_e\right) dz'$$

$$MSE^*_c(z) - MSE_c(z_B) = -\int_{z_B}^{z} \varepsilon L_v (q^* - q) dz$$

$$MSE^*_c(z) - MSE_B = -\int_{z_B}^{z} \varepsilon L_v (q^* - q) dz$$

Plugging this to B:

$$B = g \frac{MSE_B - MSE^*}{c_p K_r T} - \frac{g}{c_p K_r T} \int_{z_B}^{z} \varepsilon L_v (q^* - q) dz'$$